Question	Scheme	Marks	AOs
1(a)	$u=1+\sqrt{x}\Rightarrow x=(u-1)^2\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}u}=2(u-1)$ or $u=1+\sqrt{x}\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2}x^{-\frac{1}{2}}$	B1	1.1b
	$\int\frac{x}{1+\sqrt{x}}\mathrm{d}x=\int\frac{(u-1)^2}{u}2(u-1)\mathrm{d}u$ or $\int\frac{x}{1+\sqrt{x}}\mathrm{d}x=\int\frac{x}{u}\times 2x^{\frac{1}{2}}\mathrm{d}u=\int\frac{2x^{\frac{3}{2}}}{u}\mathrm{d}u=\int\frac{2(u-1)^3}{u}\mathrm{d}u$	M1	2.1
	$\int_0^{16}\frac{x}{1+\sqrt{x}}\mathrm{d}x=\int_1^5\frac{2(u-1)^3}{u}\mathrm{d}u$	A1	1.1b
		(3)	
(b)	$2\int\frac{u^3-3u^2+3u-1}{u}\mathrm{d}u=2\int\left(u^2-3u+3-\frac{1}{u}\right)\mathrm{d}u=\ldots$	M1	3.1a
	$=(2)\left[\frac{u^3}{3}-\frac{3u^2}{2}+3u-\ln u\right]$	A1	1.1b
	$=2\left[\frac{5^3}{3}-\frac{3(5)^2}{2}+3(5)-\ln 5-\left(\frac{1}{3}-\frac{3}{2}+3-\ln 1\right)\right]$	dM1	2.1
	$=\frac{104}{3}-2\ln 5$	A1	1.1b
		(4)	
		(7 marks)	

Notes

(a)

B1: Correct expression for $\frac{\mathrm{d}x}{\mathrm{d}u}$ or $\frac{\mathrm{d}u}{\mathrm{d}x}$ $(\text{or } u')$ or dx in terms of du or du in terms of dx

M1: Complete method using the given substitution.

This needs to be a correct method for their $\frac{\mathrm{d}x}{\mathrm{d}u}$ or $\frac{\mathrm{d}u}{\mathrm{d}x}$ leading to an integral in terms of u only (ignore any limits if present) so for each case you need to see:

$$\frac{\mathrm{d}x}{\mathrm{d}u}=\mathrm{f}(u)\rightarrow\int\frac{x}{1+\sqrt{x}}\mathrm{d}x=\int\frac{(u-1)^2}{u}\mathrm{f}(u)\mathrm{d}u$$

$\frac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{g}(x)\rightarrow\int\frac{x}{1+\sqrt{x}}\mathrm{d}x=\int\frac{x}{u}\times\frac{\mathrm{d}u}{\mathrm{g}(x)}=\int\mathrm{h}(u)\mathrm{d}u$. In this case you can condone slips with coefficients e.g. allow $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2}x^{-\frac{1}{2}}\rightarrow\int\frac{x}{1+\sqrt{x}}\mathrm{d}x=\int\frac{x}{u}\times\frac{x^{\frac{1}{2}}}{2}\mathrm{d}u=\int\mathrm{h}(u)\mathrm{d}u$

but not $\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{1}{2}x^{-\frac{1}{2}} \to \displaystyle\int \dfrac{x}{1+\sqrt{x}}\mathrm{d}x = \int \dfrac{x}{u}\times\dfrac{x^{-\frac{1}{2}}}{2}\mathrm{d}u = \int \mathrm{h}(u)\,\mathrm{d}u$

A1: All correct with correct limits and no errors. The "du" must be present but may have been omitted along the way but it must have been seen at least once before the final answer. The limits can be seen as part of the integral or stated separately.

(b)

M1: Realises the requirement to cube the bracket and divide through by u and makes progress with the integration to obtain at least 3 terms of the required form e.g. 3 from ku^3, ku^2, ku, $k\ln u$

A1: Correct integration. This mark can be scored with the "2" still outside the integral or even if it has been omitted. But if the "2" has been combined with the integrand, the integration must be correct.

dM1: Completes the process by applying their "changed" limits and subtracts the right way round **Depends on the first method mark.**

A1: Cao (Allow equivalents for $\dfrac{104}{3}$ e.g. $\dfrac{208}{6}$)

Question	Scheme	Marks	AOs
2(a)	$3^{-2}\left(1+\frac{x}{3}\right)^{-2}=3^{-2}(1+...x+...x^2)$	M1	1.1b
	$(-2)\left(\frac{x}{3}\right)$ or $\frac{(-2)(-3)}{2!}\left(\frac{x}{3}\right)^2$	M1	1.1b
	$\left(1+\frac{x}{3}\right)^{-2}=1+(-2)\left(\frac{x}{3}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x}{3}\right)^2$	A1	1.1b
	$3^{-2}\left(1+\frac{x}{3}\right)^{-2}=\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}$	A1	2.1
		(4)	

(a)

M1: Attempts a binomial expansion by taking out a factor of 3^{-2} or $\frac{1}{3^2}$ or $\frac{1}{9}$ and achieves at least the first 3 terms in their expansion. May be seen separately e.g. $\frac{1}{9}$ and $(1+...x+...x^2)$

M1: A correct method to find either the x or the x^2 term unsimplified.

Award for $(-2)(kx)$ or $\frac{(-2)(-2-1)}{2!}(kx)^2$ where $k\neq 1$. Condone invisible brackets.

A1: For a correct unsimplified or simplified expansion of $\left(1+\frac{x}{3}\right)^{-2}$ e.g. $=1+(-2)\left(\frac{x}{3}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x}{3}\right)^2-...$ or

$1-\frac{2x}{3}+\frac{x^2}{3}-...$ Do not condone missing brackets unless they are implied by subsequent work.

Condone $\left(-\frac{x}{3}\right)^2$ for $\left(\frac{x}{3}\right)^2$

Also allow this mark for 2 correct simplified terms from $\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}$ with both method marks scored.

A1: $\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}$ cao Allow terms to be listed. Ignore any extra terms. Isw once a correct **simplified** answer is seen.

<u>Direct expansion, if seen, should be marked as follows:</u>

$$\left((3+x)^{-2}=3^{-2}-2\times3^{-3}\times x+\frac{-2(-2-1)}{2!}\times3^{-4}\times x^2\right)$$

M1: For $(3+x)^{-2}=3^{-2}+3^{-3}\times\alpha x+3^{-4}\times\beta x^2$

M1: A correct method to find either the x or the x^2 term unsimplified.

Award for $(-2)\times3^{-3}x$ or $\frac{(-2)(-2-1)}{2!}\times3^{-4}x^2$. Condone invisible brackets.

A1: For a correct unsimplified or simplified expansion of $(3+x)^{-2}$ e.g. $3^{-2}-2\times3^{-3}\times x+\frac{-2(-2-1)}{2!}\times3^{-4}\times x^2$

Also award for at least 2 correct simplified terms from $\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}$ with both method marks scored.

A1: $\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}$ cao Allow terms to be listed. Ignore any extra terms. Isw once a correct **simplified** answer is seen.

Note that M0M1A1A0 is a possible mark trait in either method

<u>Note regarding a possible misread in parts (b) and (c)</u>

Some candidates are misreading $\displaystyle\int \frac{6x}{(3+x)^2}\,\mathrm{d}x$ in parts (b) and (c) as $\displaystyle\int \frac{6}{(3+x)^2}\,\mathrm{d}x$

<u>If parts (b) and (c) are consistently attempted</u> with $\displaystyle\int \frac{6}{(3+x)^2}\,\mathrm{d}x$ then we will allow the M marks in (b) **<u>only</u>**. **M1** for $x^n \to x^{n+1}$ applied to their expansion in part (a) or 6× (their expansion in part (a)) and **dM1** for substituting in 0.4 and 0.2 and subtracting either way round (may be implied).
No marks are available in part (c)

<u>MARK PARTS (b) and (c) TOGETHER</u>

(b)	$\displaystyle\int 6x \times "\left(\frac{1}{9}-\frac{2x}{27}+\frac{x^2}{27}\right)"\,\mathrm{d}x = \int\left(\frac{2x}{3}-\frac{4x^2}{9}+\frac{2x^3}{9}\right)\mathrm{d}x = ...$	**M1**	1.1b
	$\displaystyle\int\left(\frac{2x}{3}-\frac{4x^2}{9}+\frac{2x^3}{9}\right)\mathrm{d}x = \frac{x^2}{3}-\frac{4x^3}{27}+\frac{x^4}{18}$ oe	**A1**	1.1b
	$\displaystyle\left["\frac{x^2}{3}-\frac{4x^3}{27}+\frac{x^4}{18}"\right]_{0.2}^{0.4} = \left(\frac{(0.4)^2}{3}-\frac{4(0.4)^3}{27}+\frac{(0.4)^4}{18}\right)-\left(\frac{(0.2)^2}{3}-\frac{4(0.2)^3}{27}+\frac{(0.2)^4}{18}\right)$	**dM1**	3.1a
	$= \text{awrt } 0.03304$ or $\dfrac{223}{6750}$	**A1**	1.1b
		(4)	

(b)
M1: Attempts to multiply their expansion from part (a) by $6x$ or just x and attempts to integrate. Condone copying slips and slips in expanding. Look for $x^n \to x^{n+1}$ at least once having multiplied by $6x$ or x. Ignore e.g. spurious integral signs.
A1: Correct integration, simplified or unsimplified.

$$\frac{x^2}{3}-\frac{4x^3}{27}+\frac{x^4}{18} \text{ oe e.g. } \frac{1}{9}\left(3x^2-\frac{4x^3}{3}+\frac{x^4}{2}\right),\ 6\left(\frac{x^2}{18}-\frac{2x^3}{81}+\frac{x^4}{108}\right)$$

If they have extra terms they can be ignored.
Ignore e.g. spurious integral signs.
dM1: An overall problem-solving mark for

- using part (a) by integrating $6x \times$ their binomial expansion and
- substituting in 0.4 and 0.2 and subtracting either way round (may be implied)

For evidence of using the correct limits we do not expect examiners to check so allow this mark if they obtain a value with (minimal) evidence of the use of the limits 0.4 and 0.2.
This could be e.g. $\left[\mathrm{f}(x)\right]_{0.2}^{0.4} = ...$ provided the first M was scored. If the integration was correct, evidence can be taken from answer of awrt 0.0330 if limits are not seen elsewhere. **Depends on the first M mark.**
A1: awrt 0.03304 (NB allow the exact value which is $\dfrac{223}{6750} = 0.033037037...$).
Isw following a correct answer.
Note answers which use additional terms in the expansion to give a different approximation score A0
Also note that the actual value is 0.032865…

Some may use integration by parts in (b) and the following scheme should be applied.

<u>Integration by parts in (b):</u>

Either by taking $u = 6x$ and $\frac{dv}{dx} = "\left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right)"$

$$\int 6x \times "\left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right)" dx = 6x \times \left(\frac{1}{9}x - \frac{x^2}{27} + \frac{x^3}{81}\right) - 6\int\left(\frac{1}{9}x - \frac{x^2}{27} + \frac{x^3}{81}\right) dx$$

$$= 6x\left(\frac{1}{9}x - \frac{x^2}{27} + \frac{x^3}{81}\right) - \left(\frac{1}{3}x^2 - \frac{2x^3}{27} + \frac{6x^4}{324}\right)$$

M1: A full attempt at integration by parts. This requires:

$$\int 6x \times "\left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right)" dx = kx \times f(x) - k\int f(x) dx = kx \times f(x) - kg(x)$$

Where f(x) is an attempt to integrate their expansion from (a) with $x^n \to x^{n+1}$ at least once **and** g(x) is an attempt to integrate their f(x) with $x^n \to x^{n+1}$ at least once

A1: Fully correct integration. Then **dM1A1** as in the main scheme

Or by taking $u = "\left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right)"$ and $\frac{dv}{dx} = 6x$

$$\int 6x \times "\left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right)" dx = 3x^2 \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) - \int 3x^2 \times \left(-\frac{2}{27} + \frac{2x}{27}\right) dx$$

$$= 3x^2 \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) - \int\left(\frac{6x^3}{27} - \frac{6x^2}{27}\right) dx = 3x^2 \times \left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right) - \left(\frac{6x^4}{108} - \frac{6x^3}{81}\right)$$

M1: A full attempt at integration by parts. This requires:

$$\int 6x \times "\left(\frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}\right)" dx = kx^2 \times f(x) - k\int x^2 g(x) dx = kx^2 \times f(x) - kh(x)$$

Where f(x) is their expansion from (a) and g(x) is an attempt to differentiate their f(x) with $x^n \to x^{n-1}$ at least once **and** h(x) is an attempt to integrate their $x^2 g(x)$ with $x^n \to x^{n+1}$ at least once

A1: Fully correct integration. Then **dM1A1** as in the main scheme

(c)	Overall problem-solving mark (see notes)	**M1**	3.1a
	$u=3+x \Rightarrow \int_{3.2}^{3.4} \mathrm{f}(u)\,\mathrm{d}u \Rightarrow \int_{3.2}^{3.4} \frac{6(u-3)}{u^2}\,\mathrm{d}u = \int_{3.2}^{3.4} \frac{6}{u} - \frac{18}{u^2}\,\mathrm{d}u \Rightarrow \ldots\ln u + \ldots u^{-1}$	**M1**	1.1b
	$\int_{3.2}^{3.4} \frac{6(u-3)}{u^2}\,\mathrm{d}u = \int_{3.2}^{3.4} \frac{6}{u} - \frac{18}{u^2}\,\mathrm{d}u \Rightarrow 6\ln u + 18u^{-1}$	**A1**	1.1b
	$\left[6\ln u + 18u^{-1}\right]_{3.2}^{3.4} = \left(6\ln 3.4 + \frac{18}{3.4}\right) - \left(6\ln 3.2 + \frac{18}{3.2}\right) = \ldots$	**ddM1**	1.1b
	$6\ln\left(\frac{17}{16}\right) - \frac{45}{136}$ oe	**A1**	2.1
		(5)	
(c) Alt 1	Overall problem-solving mark (see notes)	**M1**	3.1a
	$\int 6x(3+x)^{-2}\,\mathrm{d}x = \frac{\ldots x}{3+x} \pm \ldots\int (3+x)^{-1}\mathrm{d}x = \frac{\ldots x}{3+x} \pm \ldots\ln(3+x)$ oe	**M1**	1.1b
	$= 6\ln(3+x) - \frac{6x}{3+x}$ oe	**A1**	1.1b
	$\left(6\ln(3+0.4) - \frac{6(0.4)}{3+0.4}\right) - \left(6\ln(3+0.2) - \frac{6(0.2)}{3+0.2}\right) = \ldots$	**ddM1**	1.1b
	$6\ln\left(\frac{17}{16}\right) - \frac{45}{136}$ oe	**A1**	2.1

(c) Alt 2	Overall problem-solving mark (see notes)	**M1**	3.1a
	$\int 6x(3+x)^{-2}\,\mathrm{d}x = \int\left(\frac{\ldots}{(3+x)} + \frac{\ldots}{(3+x)^2}\right)\mathrm{d}x = \ldots\ln(3+x) + \frac{\ldots}{3+x}$ oe	**M1**	1.1b
	$= 6\ln(3+x) + \frac{18}{3+x}$ oe	**A1**	1.1b
	$\left(6\ln(3+0.4) + \frac{18}{3+0.4}\right) - \left(6\ln(3+0.2) + \frac{18}{3+0.2}\right) = \ldots$	**ddM1**	1.1b
	$6\ln\left(\frac{17}{16}\right) - \frac{45}{136}$ oe	**A1**	2.1
			(13 marks)

Notes

(c) There are various methods which can be used

M1: An overall problem-solving mark for **all of**

- using an appropriate integration technique e.g. substitution, by parts or partial fractions – note that this may not be correct but mark positively if they have tried one of these approaches
- integrates one of their terms to a natural logarithm, e.g., $\frac{a}{3+x} \to b\ln(3+x)$ or $\frac{a}{u} \to b\ln u$
- substitutes in correct limits and subtracts either way round

M1: Integrates to achieve an expression of the required form for their chosen method

- substitution: $u = x+3 \to \pm\frac{a}{u} \pm b\ln u$ or e.g. $u = (x+3)^2 \to \pm\frac{a}{\sqrt{u}} \pm b\ln u$
- parts: $\pm a\ln(3+x) \pm \frac{bx}{3+x}$ condone missing brackets e.g. $\ldots\ln x + 3$ for $\ldots\ln(3+x)$
- partial fractions: $\pm a\ln(3+x) \pm \frac{b}{3+x}$ condone missing brackets e.g. $\ldots\ln 3 + x$ for $\ldots\ln(3+x)$

A1: Correct integration for their method e.g.

- substitution: $u = x+3 \to 6\ln u + 18u^{-1}$ or e.g. $u = (x+3)^2 \to 3\ln u + \dfrac{18}{\sqrt{u}}$
- parts: $6\ln(3+x) - \dfrac{6x}{3+x}$
- partial fractions: $6\ln(3+x) + \dfrac{18}{3+x}$ oe e.g. $3\ln(9+6x+x^2) + \dfrac{18}{3+x}$

Note that the above terms may appear "separated" but must be correct with the correct signs.
(ignore any reference to a constant of integration)
Do not condone missing brackets e.g. $6\ln x + 3$ for $6\ln(3+x)$ unless they are implied by later work.

ddM1: Substitutes in the correct limits for their integral and subtracts either way round to find a value
Depends on both previous method marks.
For evidence of using the correct limits we do not expect examiners to check so allow this mark if they obtain a value with (minimal) evidence of the use of the appropriate limits.
This could be e.g. $\left[\mathrm{f}(x)\right]_{0.2}^{0.4} = ...$ provided both previous M marks were scored.
Note that for substitution they may revert back to 3 + x and so should be using 0.4 and 0.2

A1: A full and rigorous argument leading to $6\ln\left(\dfrac{17}{16}\right) - \dfrac{45}{136}$ or exact equivalent e.g. $3\ln\left(\dfrac{289}{256}\right) - \dfrac{45}{136}$ or e.g. $-6\ln\left(\dfrac{16}{17}\right) - \dfrac{45}{136}$

The brackets are not required around the $\dfrac{17}{16}$ and allow exact equivalents e.g. allow 1.0625 or $1\dfrac{1}{16}$ but not e.g. $\dfrac{3.4}{3.2}$. The $\dfrac{45}{136}$ must be exact or an exact equivalent. Also allow e.g. $6\ln\left|\dfrac{17}{16}\right| - \dfrac{45}{136}$

Ignore spurious integral signs that may appear as part of their solution.